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U(1/1) coherent states and a path integral for the Jaynes–Cummings model

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Abstract. The path integral over the coherent states of the U(1/1) superalgebra is constructed. It is applied to the Jaynes–Cummings model whose dynamical group is U(1/1).

1. Introduction

There exists an elegant method of constructing a functional integral representation for a quantum system provided that its group of dynamical invariance is known. By the group of dynamical invariance we mean a group whose Lie algebra is a spectrum-generating algebra, so that the Hamiltonian of the system is an element of this algebra. The functional integral for a system like that is constructed as an integral over generalized Perelomov coherent states (CS) (Perelomov 1972). These states may be thought of as the set of states generated by the operators of the unitary irreducible representation (UIR) of the dynamical group acting on a certain appropriately chosen state vector. To illustrate this, some quantum-mechanical systems with the non-compact SU(1, 1) dynamical symmetry might be mentioned. These, in particular, are the hydrogen atom, the Morse oscillator and superfluid Bose systems (Gerry and Silverman 1982, Gerry 1986).

In this paper we consider the problem of constructing U(1/1) CS and the path-integral representation over those states for the simplest version of Fermi–Bose interaction—the Jaynes–Cummings (JC) model. U(1/1) algebra is known to be the 4D superalgebra with compact U(1) × U(1) even subalgebra. It happens to be the dynamical or spectrum-generating algebra for the JC model (Buzano *et al* 1989).

One may hope to use some standard path-integral methods in studying JC and related problems. One of them is based on the procedure of disentangling Pauli matrices out of the symbol of *T*-ordering operator. But it happens to be rather tiresome even for the simplest models (Kolokolov 1986). On the other hand, the standard method of integrating over fermionic and bosonic variables in the holomorphic representation, being very useful in the framework of perturbation theory, is practically useless in our case. The U(1/1) CS path integral seems to be the most appropriate one for the JC-type models.

2. JC model and U(1/1) superalgebra

The JC model is assumed to be the simplest version of matter–radiation interaction. It describes a two-level atom (or a single $\frac{1}{2}$-spin) coupled linearly with a single bosonic

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mode (Jaynes and Cummings 1963). Using the notation of b and f for the bosonic and fermionic modes, respectively, we have as usual

$$[b, b^+] = \{f, f^+\} = 1 \qquad [b, f] = [b, f^+] = 0.$$

In this notation the JC Hamiltonian may be written as

$$\hat{H}_{\mathrm{JC}} = 2\omega_1 b^+ b + 2\omega_2 f^+ f + \lambda b f^+ + b^+ f \bar{\lambda}. \tag{1}$$

An atom is here supposed to be in the eigenstates with energies $E = 2\omega_2$ and $E = 0$. The interaction constant λ may be considered in equation (1) as a Grassmann or an ordinary c-valued number. The Hamiltonian (1) is easily proved to be Hermitian. It should also be noted that instead of the operators f and f^+ one could use Pauli matrices by setting

$$\sigma_+ = f^+ \qquad \sigma_- = f \qquad \sigma_3 = f^+ f - \tfrac{1}{2}. \tag{2}$$

Let us now recall the definition of $U(1/1)$ superalgebra (Balantekin *et al* 1981). The bosonic and fermionic bilinear combinations b^+b and f^+f, entering into equation (1), generate the Lie algebras of $\mathrm{U_B}(1)$ and $\mathrm{U_F}(1)$, respectively. The Bose–Fermi bilinears bf^+ and b^+f close into the set b^+b, f^+f under anticommutation

$$\{bf^+, b^+f\} = b^+b + f^+f \qquad \{bf^+, bf^+\} = \{bf, bf\} = 0.$$

Thus, bilinear combinations bf^+ and fb^+ as the odd generators, and b^+b, f^+f as the even generators form the Lie superalgebra $\mathrm{U}(1/1)$ with the even subalgebra $\mathrm{U}(1) \times \mathrm{U}(1)$. The unitary operator $\hat{U}$ representing the $\mathrm{U}(1/1)$ supergroup action in the super Fock space formed by a tensor product of the Fock spaces of operators b and f can be presented in the form

$$\hat{U}(\omega_1, \omega_2; \theta, \bar{\theta}) = \exp(\mathrm{i}\omega_1 b^+ b + \mathrm{i}\omega_2 f^+ f + \theta b^+ f - b f^+ \bar{\theta}) \tag{3}$$

where complex Grassmann parameters θ and $\bar{\theta}$ are supposed to anticommute with fermion operators; $\omega_1, \omega_2 \in \mathbb{Q}_0$. Note that under the action of $\hat{U}(\omega_1, \omega_2; \theta, \bar{\theta})$ the operators b and f transform through each other, but the commutation relations

$$[b, b^+] = \{f, f^+\} = 1$$

remain unchanged.

Let us now express equation (1) in terms of the $\mathrm{U}(1/1)$ Casimir operator $\hat{N} = f^+f + b^+b$ and the generators $\hat{M} = \frac{1}{2}(b^+b - f^+f + 1)$, $Q = bf^+$, $Q^+ = b^+f$. The Hamiltonian (1) becomes

$$\hat{H} = \omega_2 - \omega_1 + (\omega_1 + \omega_2)\hat{N} + 2(\omega_1 - \omega_2)\hat{M} + \lambda Q + Q^+\bar{\lambda}. \tag{4}$$

The following structure equations of $\mathrm{U}(1/1)$ hold:

$$\begin{aligned} &\{Q^+, Q\} = \hat{N} \qquad [\hat{M}, \hat{Q}] = -Q \qquad [\hat{M}, Q^+] = Q^+ \\ &[\hat{M}, \hat{N}] = [Q, N] = [Q^+, N] = \{Q^+, Q^+\} = \{Q, Q\} = 0. \end{aligned} \tag{5}$$

The UIRs of the superalgebra (5) are well known (de Crombrugghe and Rittenberg 1983). In our case the Casimir operator $\hat{N}$, whose eigenvalues label the IRs of $\mathrm{U}(1/1)$, takes on non-negative integer values. So, for any integer $n > 0$ we have a 2D IR which in the basis of $(|e_1\rangle, |e_2\rangle)$ is given by

$$\begin{aligned} &Q^+|e_1\rangle = \sqrt{n}|e_2\rangle \qquad Q^+|e_2\rangle = 0 \qquad Q|e_1\rangle = 0 \qquad Q|e_2\rangle = \sqrt{n}|e_1\rangle \\ &2\hat{M}|e_1\rangle = (n-1)|e_1\rangle \qquad 2\hat{M}|e_2\rangle = (n+1)|e_2\rangle \\ &\hat{N}|e_1\rangle = n|e_1\rangle \qquad \hat{N}|e_2\rangle = n|e_2\rangle \end{aligned} \tag{6a}$$

where

$$|e_1\rangle = |n-1\rangle \times \begin{pmatrix}1\\0\end{pmatrix} \qquad |e_2\rangle = |n\rangle \times \begin{pmatrix}0\\1\end{pmatrix} \qquad b^+b|n\rangle = n|n\rangle.$$

For $n=0$ the IR of U(1/1) is 1D:

$$Q^+|e_0\rangle = Q|e_0\rangle = 0 \qquad \hat{N}|e_0\rangle = 0 \qquad |e_0\rangle = |0\rangle \times \begin{pmatrix}0\\1\end{pmatrix}. \tag{6b}$$

At this point some remarks should be made. To consider the JC Hamiltonian (4) as an element of the U(1/1) superalgebra, one must keep the parameter λ in (4) as a Grassmann number. In this case the Hamiltonian becomes an even generator of U(1/1) and can be diagonalized by the appropriate rotation in the U(1/1) superspace (Buzano *et al* 1989). But as we shall see later, in terms of the U(1/1) CS path integral there is a possibility of considering both cases, the Grassmann and c-valued λ. So, for the time being, we do not specify the parameter λ.

3. U(1/1) coherent states

We now describe U(1/1) CS as defined for the ordinary Lie groups by Perelomov (1972) and generalized for Lie supergroups by Bars and Günaydin (1983). These states are obtained by operating on a vector of the U(1/1) IRs by a group element in the form of equation (3). Consider a 'lowest state' $|e_1\rangle$ that transforms irreducibly under U(1) × U(1) and is annihilated by the operator Q. We define the U(1/1) coherent state $|\theta; n\rangle$ labelled by the Grassmann parameter θ that belongs to the coset space U(1/1)/U(1) × U(1) as

$$|\theta; n\rangle = \exp(-\theta Q^+)|e_1\rangle = |e_1\rangle - \sqrt{n}\,\theta|e_2\rangle. \tag{7}$$

The overlap of two states $|\theta; n\rangle$ and $|\theta'; n\rangle$ is given as

$$\langle\theta'; n|\theta; n\rangle = 1 + n\bar{\theta}'\theta = \exp(n\bar{\theta}'\theta).$$

An important property of these states is that they satisfy the completeness relation

$$\hat{I} = |e_1\rangle\langle e_1| + |e_2\rangle\langle e_2| = \int \exp(n\theta\bar{\theta})|\theta; n\rangle\langle\theta; n| \frac{\mathrm{d}\bar{\theta}\,\mathrm{d}\theta}{n} \tag{8}$$

where $\hat{I}$ denotes unity in the representation n. We use the following definitions:

$$\int \mathrm{d}\theta = \int \mathrm{d}\bar{\theta} = 0 \qquad \int \theta\,\mathrm{d}\theta = \int \bar{\theta}\,\mathrm{d}\bar{\theta} = 1.$$

Note that the CS (7) depend on the representation index n. For every value of n there exists an overcomplete set of CS $|\theta; n\rangle$. For $n=1$ the CS (7) coincide with the so-called fermionic coherent states introduced by Ohnuki and Kashiwa (1978). It is quite natural due to the fact that the operators Q^+ and Q at $n=1$ become the usual fermionic operators, as follows from equations (6*a*).

It is straightforward to see that

$$\int \langle\theta; n|e_i\rangle\langle e_j|\theta; n\rangle \exp(-n\theta\bar{\theta}) \frac{\mathrm{d}\theta\,\mathrm{d}\bar{\theta}}{n} = \delta_{ij}. \tag{9}$$

By virtue of equation (9) for any operator $\hat{F}$ acting in the 2D space spanned by $|e_1\rangle$ and $|e_2\rangle$ we have

$$\mathrm{Sp}\,\hat{F}=\sum_{ij}\langle e_i|\hat{F}|e_j\rangle\delta_{ij}=\int \exp(-n\theta\bar{\theta})\langle\theta;\,n|\hat{F}|\theta;\,n\rangle\frac{\mathrm{d}\theta\,\mathrm{d}\bar{\theta}}{n}. \tag{10}$$

For definiteness we assume that any Grassmann number is commutable with the states $|e_1\rangle$ and $\langle e_1|$:

$$\theta|e_1\rangle=|e_1\rangle\theta \qquad \theta\langle e_1|=\langle e_1|\theta \qquad \text{(the same for } \bar{\theta}\text{)}.$$

For further use we present the averages over U(1/1) CS of the operators entering into equation (4):

$$\begin{aligned}
&\frac{\langle\theta;\,n|Q^+|\theta;\,n\rangle}{\langle\theta;\,n|\theta;\,n\rangle}=n\bar{\theta} \qquad &&\frac{\langle\theta;\,n|Q|\theta;\,n\rangle}{\langle\theta;\,n|\theta;\,n\rangle}=n\theta\\
&\frac{\langle\theta;\,n|\hat{N}|\theta;\,n\rangle}{\langle\theta;\,n|\theta;\,n\rangle}=n \qquad &&\frac{\langle\theta;\,n|2\hat{M}|\theta;\,n\rangle}{\langle\theta;\,n|\theta;\,n\rangle}=n-1+2n\bar{\theta}\theta.
\end{aligned} \tag{11}$$

4. Path integral

We now consider the path integral over U(1/1) CS for the JC partition function

$$Z=\mathrm{Sp}\exp(-\beta\hat{H}).$$

On account of equation (10) we have

$$Z=1+\sum_{n=1}\int \exp(-n\theta^{(n)}\bar{\theta}^{(n)})\langle\theta^{(n)};\,n|\exp(-\beta\hat{H})|\theta^{(n)};\,n\rangle\frac{\mathrm{d}\theta^{(n)}\,\mathrm{d}\bar{\theta}^{(n)}}{n} \tag{12}$$

where the first term is nothing but

$$1=\langle e_0|\exp(-\beta\hat{H})|e_0\rangle.$$

Defining ε as β/N and using equation (8) we write in the usual manner

$$\begin{aligned}
Z=1+\sum_{n=1}\int &\exp(-n\theta\bar{\theta})\frac{\mathrm{d}\theta\,\mathrm{d}\bar{\theta}}{n}\langle\theta;\,n|\xi_N;\,n\rangle\langle\xi_N;\,n|\exp(-\varepsilon\hat{H})|\xi_{N-1};\,n\rangle\\
&\times\langle\xi_{N-1}|\ldots\exp(-\varepsilon\hat{H})|\xi_0;\,n\rangle\langle\xi_0;\,n|\theta;\,n\rangle\\
&\times\exp\left(-n\sum_{k=0}^{N}\bar{\xi}_k\xi_k\right)\prod_{k=0}^{N}(\mathrm{d}\bar{\xi}_k\,\mathrm{d}\xi_k/n).
\end{aligned} \tag{13}$$

It is to be kept in mind that all Grassmann variables of integration $\bar{\theta}$, θ, $\bar{\xi}_N$, $\xi_N,\ldots\bar{\xi}_0$, ξ_0 are supposed to be labelled by the representation index n, as is done in equation (12). To simplify the notation, we omit this dependence for the moment. For small ε we have

$$\begin{aligned}
\langle\xi_i;\,n|&\exp(-\varepsilon\hat{H})|\xi_j;\,n\rangle\\
&\approx\langle\xi_i;\,n|(1-\varepsilon\hat{H})|\xi_j;\,n\rangle\\
&=\langle\xi_i;\,n|\xi_j;\,n\rangle(1-\varepsilon H(\bar{\xi}_i,\xi_j))\\
&\approx\langle\xi_i;\,n|\xi_j;\,n\rangle\exp(-\varepsilon H(\bar{\xi}_i,\xi_j))
\end{aligned}$$

where

$$H(\bar{\xi}_i, \xi_j) = \frac{\langle \xi_i; n|\hat{H}|\xi_j; n\rangle}{\langle \xi_i; n|\xi_j; n\rangle}.$$

The integration over θ, $\bar{\theta}$ in accordance with equation (10) yields

$$\begin{aligned}\int \exp(-n\theta\bar{\theta})\langle\theta; n|\xi_N; n\rangle\langle\xi_0; n|\theta; n\rangle \frac{\mathrm{d}\theta\,\mathrm{d}\bar{\theta}}{n} &= \mathrm{Sp}(|\xi_N; n\rangle\langle\xi_0; n|)\\ &= \mathrm{Sp}(|e_1\rangle\langle e_1| + n\xi_N|e_2\rangle\langle e_2|\bar{\xi}_0)\\ &= 1 + n\xi_N\bar{\xi}_0 = \exp(-n\bar{\xi}_0\xi_N)\end{aligned}$$

so that

$$Z = 1 + \sum_{n=1} \int \prod_{k=0}^{N} (\mathrm{d}\bar{\xi}_k\,\mathrm{d}\xi_k/n) \times \exp\left(-n\sum_{k=1}^{N} \bar{\xi}_k(\xi_k - \xi_{k-1}) - \varepsilon \sum_{k=1}^{N} H(\bar{\xi}_k; \xi_{k-1}) - n\bar{\xi}_0\xi_0 - n\bar{\xi}_0\xi_N\right). \tag{14}$$

The integration over $\bar{\xi}_0$, ξ_0 in (14) can be carried out explicitly to yield

$$Z = 1 + \sum_{n=1} \int \prod_{k=1}^{N} (\mathrm{d}\bar{\xi}_k\,\mathrm{d}\xi_k/n) \times \exp\left(-n\sum_{k=1}^{N} \bar{\xi}_k(\xi_k - \xi_{k-1}) - \varepsilon \sum_{k=1}^{N} H(\bar{\xi}_k; \xi_{k-1})\right)\Bigg|_{\xi_N = -\xi_0}.$$

In the continuous limit this may be written as

$$Z = 1 + \sum_{n=1} \int_{\xi^{(n)}(0) = -\xi^{(n)}(\beta)} D\bar{\xi}^{(n)}(t) D\xi^{(n)}(t) \times \exp\left(-n\int_0^\beta \bar{\xi}^{(n)}(t)\dot{\xi}^{(n)}(t)\,\mathrm{d}t - \int H(\bar{\xi}^{(n)}; \xi^{(n)})\,\mathrm{d}t\right) \tag{15}$$

where we restore the n-dependence of the ξ. Note that each of the integrals over $\bar{\xi}^{(n)}$, $\xi^{(n)}$ is normalized by the condition

$$\int_{\xi^{(n)}(0) = -\xi^{(n)}(\beta)} D\bar{\xi}^{(n)}(t) D\xi^{(n)}(t) \exp\left(-n\int_0^\beta \bar{\xi}^{(n)}(t)\dot{\xi}^{(n)}(t)\,\mathrm{d}t\right) = 2 = \mathrm{Sp}\,\hat{I}.$$

For the JC model, making use of equation (11) we obtain

$$Z_{\mathrm{JC}} = 1 + \sum_{n=1} \int_{\xi^{(n)}(0) = -\xi^{(n)}(\beta)} D\bar{\xi}^{(n)}(t) D\xi^{(n)}(t) \times \exp\left(-\int_0^\beta \bar{\xi}^{(n)}(t)\dot{\xi}^{(n)}(t)\,\mathrm{d}t - 2(\omega_2 - \omega_1)\beta - 2n\omega_1\beta - 2(\omega_1 - \omega_2) \times \int_0^\beta \bar{\xi}^{(n)}\xi^{(n)}\,\mathrm{d}t - \sqrt{n}\int_0^\beta (\bar{\xi}^{(n)}\bar{\lambda} + \lambda\xi^{(n)})\,\mathrm{d}t\right) \tag{16}$$

where the change $\xi^{(n)} \Rightarrow \xi^{(n)}/\sqrt{n}$, $\bar{\xi}^{(n)} \Rightarrow \bar{\xi}^{(n)}/\sqrt{n}$ has been made.

If one considers λ as a Grassmann parameter, the integration in (16) can be easily performed in the usual fashion by making a simple shift of the integration variables $\bar{\xi}^{(n)}(t)$ and $\xi^{(n)}(t)$. This leads to the known result by Buzano *et al* (1989), obtained by direct diagonlization of the $\hat{H}_{\rm JC}$ (4). In the case of c-valued λ the path integral in (16) over $\bar{\xi}^{(n)}(t)$, $\xi^{(n)}(t)$ may be easily recognized as a partition function for a single $\frac{1}{2}$-spin coupled with a constant field. To be more precise, let us consider the Hamiltonian

$$\hat{H}_0 = 2\Omega\sigma_3 + \bar{g}\sigma_+ + g\sigma_-$$

Ω and g being real and complex constants, respectively. The eigenvalues of the $\hat{H}_0$ are $+(\Omega^2+g\bar{g})^{1/2}$ and $-(\Omega^2+g\bar{g})^{1/2}$, so that

$$Z_0 = \mathrm{Sp}\exp(-\beta\hat{H}_0) = 2\cosh\beta\sqrt{\Omega^2+g\bar{g}}.$$

On the other hand, through the fermionic operators (2) $\hat{H}_0$ can be presented as

$$\hat{H}_0 = 2\Omega(f^+f-\tfrac{1}{2}) + gf + \bar{g}f^+. \tag{17}$$

As a result of equation (17) the fermion path-integral representation for the partition function Z_0 is obvious (see, for example, Soper 1978)

$$Z_0 = \int_{\xi(0)=-\xi(\beta)} D\bar{\xi}D\xi \exp\left(-\int_0^\beta \bar{\xi}\dot{\xi}\,\mathrm{d}t - 2\Omega\int_0^\beta \bar{\xi}\xi\,\mathrm{d}t - \int_0^\beta (g\xi+\bar{g}\bar{\xi})\,\mathrm{d}t + \Omega\beta\right) \tag{18}$$

$$Z_0(\Omega = g = 0) = 2.$$

Note that equation (18) follows from our approach as a particular case. Namely, it is a path-integral representation for the partition function Z_0 over CS (7) provided $n=1$. Comparing equations (16) and (18) we find

$$Z_{\rm JC} = 1 + 2\sum_{n=1} \exp(-2n\omega_1\beta + \beta(\omega_1-\omega_2))\cosh\beta\sqrt{(\omega_1-\omega_2)^2+\lambda\bar{\lambda}n}. \tag{19}$$

This enables us to conclude that the eigenvalues of the JC Hamiltonian are

$$E_0 = 0 \qquad E^{\pm}_{n\geqslant 1} = 2\omega_1 n + \omega_2 - \omega_1 \pm \sqrt{(\omega_1-\omega_2)^2+\lambda\bar{\lambda}n}$$

which are the correct expressions (see, for example, Agarwal and Puri 1986).

5. Conclusions

The path-integral representation over U(1/1) CS, which we have considered for the JC model, may also be applied to various nonlinear JC-type models, provided their Hamiltonians belong to the U(1/1) enveloping algebra. For example, one can easily obtain the path-integral representation for the partition function $Z = \exp(-\beta\hat{H})$ where

$$\hat{H} = \omega_1\hat{N} + \omega_2\hat{M} + \lambda Q^+QQ^+ + QQ^+Q\bar{\lambda}.$$

On the other hand, to represent in a similar way the partition function of the 'dressed' JC or Rabi model, one has to deal with the Osp(2/2) CS. The 8D non-compact Osp(2/2) superalgebra happens to be the spectrum-generating algebra for the 'dressed' JC model (Buzano *et al* 1989). To label Osp(2/2) CS, one has to use Grassmann and c-valued parameters simultaneously and the corresponding path integral turns out to be a more complicated construction. This and related problems will be discussed elsewhere.

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